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WEAKENING OF THE SIGN-DEFINITENESS CONDITION FOR THE DERIVATIVE IN SOME THEOREMS OF LYAPUNOV'S SECOND METHOD†

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Problems relating to the analysis of instability and asymptotic stability are considered for non-steady systems of ordinary differential equations, solved for the derivative. It is assumed that the right-hand sides of the system converge uniformly as the time increases without limit, tending to certain functions of the phase variables. Propositions are proved analogous to those of Lyapunov's second method [1–7] for steady systems, but the condition that the derivative of the Lyapunov function be sign-definite is relaxed. Instead, the derivative is required to be of constant sign, and a certain algebraic condition, which may always be verified directly, is imposed on the Lyapunov function. © 2004 Elsevier Ltd. All rights reserved.

1. A CONDITION THAT POINTS OF ω-LIMIT SETS MUST SATISFY

Consider the system

$$\dot{x}_i = \upsilon_i(t, x), \quad 0 \le t < \infty, \quad x \in B_{\varepsilon_0} \in \mathbf{R}^n; \quad \upsilon_i(t, x) \in C_{tx}^{0,1}(\mathbf{R}_{\{t\}}^+ \times B_{\varepsilon_0}) \tag{1.1}$$

where B_{ε_0} is a certain open sphere in $\mathbf{R}^n$ with centre at the point $x_0 = 0$ and radius ε_0.

Suppose the functions $\upsilon_i(t,x)$ $(i = 1, \dots, n)$ in system (1.1) converge as $t \to \infty$ uniformly in the domain $\bar{B}_{\varepsilon_0}$ to the functions $\upsilon_i^*(x)$:

$$\upsilon_i(t, x) \underset{t \to \infty}{\overset{}{\rightrightarrows}}_{B_{\varepsilon_0}} \upsilon_i^*(x), \quad \upsilon_i^*(x) \in C^m(B_{\varepsilon_0}), \quad m \ge 0, \quad i = 1, 2, \dots, n \tag{1.2}$$

We shall say that such systems belong to class $\mathcal{K}$. For every system (1.1) of class $\mathcal{K}$, in the domain $(t, x) \subset \mathbf{R}_{\{t\}}^+ \times B_{\varepsilon_0}$ conditions hold for the existence and uniqueness of solutions and their continuous dependence on the time t (this property will be repeatedly needed below) and on the initial data [3].

Lemma 1. Suppose that, for some trajectory $x(t; \hat{t}, \hat{x})$, $0 < \hat{t}, \hat{x} \in B_{\varepsilon_0}$ of a system (1.1) of class $\mathcal{K}$, one can define a non-empty ω-limit set $\pi(\hat{t}, \hat{x}) \subset B_{\varepsilon_0}$ which is a subset of a level set of a function $F(x) \in C^1(B_{\varepsilon_0})$.

Then

$$\pi(\hat{t}, \hat{x}) \subset \left\{ x\colon \lim_{t \to \infty} \mathcal{D}[F]\upsilon(t, x) \overset{\text{def}}{=} \mathcal{D}_*^{(1)}[F](x) = 0 \right\}$$

Proof. By virtue of the convergence (1.2) and the condition $F(x) \in C^1(B_{\varepsilon_0})$, the function

$$\lim_{t \to \infty} dF(x)/dt|_{(t, x)} = \mathcal{D}_*^{(1)}F(x)$$

is well defined in the domain B_{ε_0}.

We will show that under the assumptions of Lemma 1 the set $\pi(\hat{t}, \hat{x})$ is a subset of the zero level of this function.

Since a limit set is always closed and the sphere B_{ε_0} is an open set, it follows that for every number ε_0 there is a number $\tilde{\varepsilon}_0 < \varepsilon_0$ such that

$$\pi(\hat{t}, \hat{x}) \subset B_{\tilde{\varepsilon}_0} \subset \bar{B}_{\tilde{\varepsilon}_0} \subset B_{\varepsilon_0}$$

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A number $M > 0$ exists such that

$$|v_i(t,x)| < M, \quad \forall (t,x): t \geq 0, \quad x \in B_{\tilde{\varepsilon}_0}, \quad i = 1, \ldots, n \tag{1.3}$$

Indeed, suppose the contrary, that is,

$$\exists i^*: \exists t_n \geq 0, \quad \exists x_n \in B_{\tilde{\varepsilon}_0}: |v_{i*}(t_n, x_n)| > N, \forall N \tag{1.4}$$

The sequence $\{x_n\}$ in the compact set $\bar{B}_{\tilde{\varepsilon}_0}$ always contains a convergent subsequence, so that we may assume, without loss of generality, that $\{x_n\} \to x^{(0)}, x^{(0)} \in \bar{B}_{\tilde{\varepsilon}_0}$.

There are two possibilities for the sequence $\{t_n\}$:

(1) the sequence $\{t_n\}$ is bounded above by a number $T > 0$: $t_n < T$, $\forall n$;

(2) the sequence $\{t_n\}$ contains a subsequence $\{\tilde{t}_m\} \to \infty$, $\tilde{t}_{k+1} > \tilde{t}_k$.

In the first case, assumption (1.4) contradicts the fact that the function $v_{i*}(t,x)$, being by assumption continuous in the compact set $\{t, x: 0 \leq t \leq T, x \in \bar{B}_{\tilde{\varepsilon}_0}$, is uniformly bounded there. Consequently, assumption (1.4) may only hold if the second possibility holds.

Since $x^{(0)} \in \bar{B}_{\tilde{\varepsilon}_0}$, it follows from the condition of uniform convergence $v_{i*}(t,x) \underset{t\to\infty}{\rightrightarrows} v_{i*}^*(x)$ in the domain B_{ε_0} that

$$\begin{aligned} &\forall \varepsilon\ \exists \tau(\varepsilon, x^{(0)}), \delta(\varepsilon, x^{(0)}): \\ &|v_{i*}(t,x) - v_{i*}^*(x^{(0)})| < \varepsilon\ \forall x: |x - x^{(0)}| < \delta,\ t > \tau(\varepsilon, x^{(0)}) \end{aligned} \tag{1.5}$$

Fix some $\varepsilon = \hat{\varepsilon} > 0$.

Assuming that the second possibility holds, we infer from the convergence condition $\{x_n\}_{n\to\infty} \to x^{(0)}$ that

$$\exists N_0: \forall m > N_0: |\tilde{x}_m - x^{(0)}| < \delta(\hat{\varepsilon}, x^{(0)}), \quad \tilde{t}_m > \tau(\hat{\varepsilon}, x^{(0)}) \tag{1.6}$$

where $\{\tilde{x}_m\}$ is the subsequence of $\{x_n\}$ corresponding to the subsequence $\{\tilde{t}_m\}$ of $\{t_n\}$. Reasoning now from condition (1.5) with $\varepsilon = \hat{\varepsilon}$, in view of condition (1.6) and the boundedness in $\bar{B}_{\tilde{\varepsilon}_0}$ of the function $v_{i*}^* \in C(B_{\varepsilon_0})$, we obtain a contradiction to assumption (1.4), proving the validity of the estimate (1.3).

Now suppose Lemma 1 is false:

$$\exists x^* \in \pi(\hat{t}, \hat{x}); \quad |x^*| < \tilde{\varepsilon}_0: \mathfrak{D}_*^{(1)} F(x^*) \neq 0 \tag{1.7}$$

Let $\mathfrak{D}_*^{(1)} F(x^*) = 2\alpha$; without loss of generality, we shall assume that $\alpha > 0$.

It follows immediately from the fact that system (1.1) is of class $\mathcal{K}$, in view of the continuity of the function $F(x)$ in the domain B_{ε_0}, that the convergence of the function

$$\dot{F}(t,x) = \frac{\partial F}{\partial x_k} v_k(t,x)$$

as $t \to \infty$ to the function $\mathfrak{D}_*^{(1)}[F(x^*)](x)$ is uniform in the domain $B_{\tilde{\varepsilon}_0}$. Therefore,

$$\begin{aligned} &\exists T_0 = T_0(x^*, \alpha) > 0, \quad \delta_0 = \delta_0(x^*, \alpha), \quad |x^*| + \delta_0 < \tilde{\varepsilon}_0: \\ &|\dot{F}(t,x) - \mathfrak{D}_*^{(1)} F(x^*)| < \alpha\ \forall x: |x - x^*| < \delta_0, \quad \forall t > T_0 \end{aligned} \tag{1.8}$$

Hence we have

$$\dot{F}(t,x) > \alpha\ \forall x: |x - x^*| < \delta_0, \quad \forall t > T_0 \tag{1.9}$$

By assumption, x^* is a limit point for the trajectory $x(t;\ \hat{t}, \hat{x})$, and hence an increasing sequence $\{t\} \to \infty$: $\{x(t_n;\ \hat{t}, \hat{x})\} \to x^*$ exists. Thus, for the numbers δ_0 and T_0 of (1.9) we obtain

$$\exists N_1 = N_1(\delta_0, T_0): |x^* - x(t_n;\ \hat{t}, \hat{x})| < \delta_0/2; \quad t_n > T_0, \quad \forall n > N_1 \tag{1.10}$$

Let us now assume that for some number $n_0 > N_1$

$$|x^* - x(t_{n_0} + t;\ \hat{t}, \hat{x})| < (3/4)\delta_0, \quad \forall t > 0 \tag{1.11}$$

Then estimate (1.9) is valid along the trajectory $x(t;\ \hat{t},\ \hat{x})$ (the trajectory is by assumption defined throughout the interval $\mathbf{R}^+_{\{t\}}$) beginning from the time t_{n_0}. We therefore have

$$F(x)|_{x = x(t_{n_0}+t;\ \hat{t},\ \hat{x})} > F(x)|_{x = x(t_{n_0}+t;\ \hat{t},\ \hat{x})} + \alpha t, \quad \forall t > 0$$

But this, together with the assumption (1.11), as well as the condition $|x^*| + \delta_0 < \tilde{\varepsilon}_0$ in assumption (1.8), contradicts the boundedness in the compact set $\bar{B}_{\varepsilon_0}$ of the function $F(x)$, which is continuous in $\bar{B}_{\tilde{\varepsilon}_0} \subset \bar{B}_{\varepsilon_0}$. Hence assumption (1.11) is false. Thus, for all $n > N_1$ a finite value of $\tau_n > 0$ exists such that the trajectory $x(t;\ \hat{t},\ \hat{x})$, emanating at time $t = t_n$ from the point $x(t_n;\ \hat{t},\ \hat{x})$, first intersects the sphere $|x - x^*| = (3/4)\delta_0$ at the time $t_n + \tau_n$:

$$\begin{aligned} &\exists \tau_n < \infty\ \left|x(t_n + \tau_n;\ \hat{t},\ \hat{x}) - x^*\right| = (3/4)\delta_0, \quad \forall n > N_1; \\ &\left|x(t_n + t;\ \hat{t},\ \hat{x}) - x^*\right| < (3/4)\delta_0\colon\ 0 \le t < \tau_n \end{aligned} \tag{1.12}$$

Next, taking into account that $\{x\colon |x - x^*| \leqslant (3/4)\delta_0\} \subset B_{\tilde{\varepsilon}_0}$ (see the corresponding condition in (1.8)) we deduce by virtue of the estimate (1.3) that for any point x of the sphere $|x - x^*| = \delta_0/2$ and any time $t > 0$, the time in which a trajectory of system (1.1), emanating at time t from the point x, will succeed in reaching the sphere $|x - x^*| = (3/4)\delta_0$, is no less than a certain quantity $\tau_0(x^*) = (1/4)\delta_0 M\sqrt{n} > 0$.

By conditions (1.10) and (1.12), this implies that

$$\tau_n \ge \tau_0, \quad \forall n > N_1 \tag{1.13}$$

It now follows from the condition $x^* \in \pi(\hat{t}, \hat{x}) \subset B_{\tilde{\varepsilon}_0}$, the continuity of the function $F(x)$ in the domain $B_{\tilde{\varepsilon}_0}$, and the convergence $\{x(t_n;\ \hat{t},\ \hat{x})\}_{n\to\infty} \to x^*$, that

$$\exists N_2\colon \left|F(x(t_n;\ \hat{t},\ \hat{x})) - F(x^*)\right| < \alpha\tau_0/2, \quad \forall n > N_2 \tag{1.14}$$

The numbers α and τ_0 were defined above.

As a result, from conditions (1.10), (1.2), (1.9), (1.13) and (1.14), we obtain

$$\begin{aligned} F(x(t_n + \tau_n;\ \hat{t},\ \hat{x})) &= F(x(t_n;\ \hat{t},\ \hat{x})) + \int_{t_n}^{t_n+\tau_n} \dot{F}(\tau, x(\tau;\ \hat{t},\ \hat{x}))d\tau > \\ &> F(x(t_n;\ \hat{t},\ \hat{x})) + \alpha\tau_0 > F(x^*) + \alpha\tau_0/2, \quad \forall n > N_0 = \max\{N_1, N_2\} \end{aligned} \tag{1.15}$$

On the other hand, since the sequence $\{t_n\} \to \infty$, while for any $n > N_0$ the number $\tau_n > \tau_0 > 0$ is finite, we can extract from the sequence $\{t_n + \tau_n\}$ a monotonically increasing subsequence that tends to infinity:

$$\exists n(m)\colon \mathbf{N} \to \mathbf{N}\colon \tilde{t}_m = t_{n(m)} + \tau_{n(m)}\colon \tilde{t}_{k+1} > \tilde{t}_k; \quad \{\tilde{t}_m\} \to \infty$$

In turn, all the points $x(t_{n(m)} + \tau_{n(m)};\ \hat{t}, \hat{x}) = x(\tilde{t}_m;\ \hat{t}, \hat{x})$ lie, by construction, on the sphere $|x - x^*| = (3/4)\delta_0$, and therefore we can extract from the sequence $x(\tilde{t}_m;\ \hat{t}, \hat{x})$ a subsequence $\{x(\tilde{\tilde{t}}_s;\ \hat{t}, \hat{x})\}$ which converges to some point x^{**}: $|x^* - x^{**}| = (3/4)/\delta_0$. Now, for every s, $\{x(\tilde{t}_s;\ \hat{t}, \hat{x})\}$ is a point of the trajectory $\{x(t;\ \hat{t}, \hat{x})\}$, and the sequence $\{\tilde{\tilde{t}}_s\} \to \infty$, as a subsequence of $\{\tilde{t}_m\}$, tends to $+\infty$; hence

$$x^{**} \in \pi(\hat{t}, \hat{x})$$

By the assumptions of Lemma 1, we may therefore conclude that

$$F(x^*) = F(x^{**}) \tag{1.16}$$

At the same time, by virtue of the continuity of the function F in the domain $B_{\tilde{\varepsilon}_0}$, the condition $|x^*| + \delta_0 < \tilde{\varepsilon}_0$ in (1.8), the first condition (1.12) and the estimate (1.15), we conclude that the following estimate holds at the point x^{**}, as a partial limit of the sequence $\{x(t_n + \tau_n;\ \hat{t}, \hat{x})\}$

$$F(x^{**}) \ge F(x^*) + \alpha\tau_0/2, \quad \alpha\tau_0/2 > 0$$

But this contradicts equality (1.16).

Thus, assumption (1.7) is false, and this proves Lemma 1.

Suppose now that, under the assumptions of Lemma 1, $F(x)$ is a function of class C^2 in the domain B_{ε_0}. Then, if $m \geqslant 1$, we have $\mathfrak{D}_*^{(1)}[F](x) \in C^1(B_{\varepsilon_0})$. Thus, Lemma 1 may be applied in this case twice in succession: first to the function $F(x)$ and then to the function $\mathfrak{D}_*^{(1)}[F](x)$.

In the case of a system (1.1) of class $\mathcal{K}$ and a function $F(x) \in C^{m+1}(B_{\varepsilon_0})$, Lemma 1 may obviously be applied successively $m + 1$ times. This follows immediately from the formulation of Lemma 1 and the fact that, in that case, the condition

$$\Phi(x) \in C^p(B_{\varepsilon_0}), \quad \forall p : 2 \le p \le m+1$$

implies

$$\mathfrak{D}_*^{(1)}[\Phi](x) \in C^{p-1}(B_{\varepsilon_0})$$

Thus, the following proposition holds.

Corollary 1. Suppose that, under the assumptions of Lemma 1, $F(x)$ is a function of class $C^{m+1}(B_{\varepsilon_0})$. Then

$$\pi(\hat{t}, \hat{x}) \subset \{x : \mathfrak{D}_*^{(1)}[F](x) = 0, \mathfrak{D}_*^{(2)}[F](x) = 0, \ldots, \mathfrak{D}_*^{(m+1)}[F](x) = 0\}$$

where

$$\mathfrak{D}_*^{(p+1)}[F](x) = \mathfrak{D}_*^{(1)}[\mathfrak{D}_*^{(p)}[F]](x)$$

the function $\mathfrak{D}_*^{(1)}[F](x)$ being defined as in the statement of Lemma 1.

We now note that any dynamical system $\dot{x} = \omega(x), x \in \mathbf{R}^n$ with autonomous field of velocities of class $C^m(B_{\varepsilon_0})$ is necessarily a system of class $\mathcal{K}$. Therefore, obviously, all of the previous exposition automatically carries over to the autonomous case. Corollary 1 of Lemma 1 becomes

Corollary 2. Suppose that, for some trajectory $g^t(\hat{x})$ of an autonomous system (1.1) defined by a field of class $C^m(B_{\varepsilon_0})$, an ω-limit set $\pi(\hat{x})$ exists, all of whose points lie in the open sphere B_{ε_0} and to a level set of a function $F(x) \in C^{m+1}(B_{\varepsilon_0})$. Then

$$\pi(\hat{x}) \subset \{x: \mathfrak{D}^k F(x) = 0, k = 1, \ldots, m+1\}$$

where $\mathfrak{D}^p F(x)$ is the pth Lie derivative of $F(x)$ along trajectories of the system (1.1) under consideration. This elementary fact may obviously be proved without the use of Lemma 1.

Lemma 2. Let J be an invariant set of system (1.1) such that $\bar{J} \in \mathbf{R}^+_{\{t\}} \times B_{\varepsilon_0}$ is a subset of a level set of a function $F(t,x) \in C^{m+1}(\mathbf{R}^+_{\{t\}} \times B_{\varepsilon_0})$. Then the set $\bar{J}$ is a subset of the common zero level set of the functions $\mathfrak{D}^k F(t, x)$ $(k = 1, \ldots, m + 1)$, $J \in \{(t, x): \mathfrak{D}^k F(t, x) = 0, k = 1, 2, \ldots, m + 1\}$, where $\mathfrak{D}^p F(t,x)$ is the total pth derivative of the function $F(t,x)$ along trajectories of Eqs (1.1).

Proof. By the condition $\bar{J} \in \mathbf{R}^+_{\{t\}} \times B_{\varepsilon_0}$, we have

$$\exists \tilde{\varepsilon}_0 < \varepsilon_0: \bar{J} \subset (\mathbf{R}^+_{\{t\}} \times B_{\tilde{\varepsilon}_0}) \subset (\mathbf{R}^+_{\{t\}} \times \bar{B}_{\varepsilon_0}) \subset (\mathbf{R}^+_{\{t\}} \times B_{\varepsilon_0})$$

By the conditions

$$\omega_i \in C^m(\mathbf{R}^+_{\{t\}} \times B_{\varepsilon_0}), \quad F(t, x) \in C^{m+1}(\mathbf{R}^+_{\{t\}} \times B_{\varepsilon_0})$$

all the functions $\mathfrak{D}^p F(t,x)$ $(p = 1, \ldots, m + 1)$ are well defined in the domain $\mathbf{R}^+_{\{t\}} \times B_{\varepsilon_0}$. We will show that the set $\bar{J}$ is a subset of their common zero level.

Let $(\hat{t}, \hat{x})$ be an arbitrary point of the set J. Consider a trajectory $x(t;\ \hat{t}, \hat{x})$ of system (1.17) and a sequence $\{t_k\}_{k \to \infty} \to \hat{t}$: $\forall k \hat{t} < t_{k+1} < t_k$.

By assumption, the point $(t, x(t;\ \hat{t}, \hat{x}))$, $\forall t > \hat{t}$ belongs to J, and therefore

$$F(t_k, x(t_k;\ \hat{t}, \hat{x})) = F(\hat{t}, \hat{x}), \quad \forall k = 1, 2, \ldots$$

Hence, noting that

$$F(t_k, x(t_k;\ \hat{t}, \hat{x})) = F(\hat{t}, \hat{x}) + \int^{t_k} \dot{F}(\tau, x(\tau;\ \hat{t}, \hat{x}))d\tau$$

and that, as a consequence of the condition $\bar{J} \subset (\mathbf{R}^+_{\{t\}} \times B_{\varepsilon_0})$, the whole semi-trajectory $(\tau, x(\tau, \hat{t}, \hat{x}))$, $\tau \geqslant \hat{t}$ lies in the continuity domain of the function $\dot{F}(t, x)$, we obtain

$$\exists t_k^{(1)}: \hat{t} < t_k^{(1)} < t_k: \dot{F}(t, x)\big|_{t = (t_k^{(1)};\ x = x(t_k^{(1)};\ \hat{t}, \hat{x}))} = 0$$

Next, for functions $\upsilon_i(t,x)$ continuous in the domain $\mathbf{R}^+_{\{t\}} \times B_{\varepsilon_0}$, we conclude that they are bounded in the compact set $\{(t,x): \hat{t} \leqslant t \leqslant t_1, x \in \bar{B}_{\bar{\varepsilon}_0}\}$:

$$\exists M_0: |\upsilon_i(t, x)| < M_0, \quad \forall t, \quad \hat{t} \le t \le t_1, \quad \forall x \in \bar{B}_{\bar{\varepsilon}_0}, \quad i = 1, \dots, n$$

Hence, in view of the conditions

$$\hat{t} < t_k^{(1)} < t_k \le t_1 \quad \text{and} \quad (t_k^{(1)}, x(t_k^{(1)}; \hat{t}, \hat{x})) \in J \in \bar{B}_{\bar{\varepsilon}_0} \times \bar{\mathbf{R}}^+_{\{}$$

we obtain

$$|\hat{x} - x(t_k^{(1)}; \hat{t}, \hat{x})| < \sqrt{n} M_0 (t_k^{(1)} - \hat{t})$$

This last inequality, together with the conditions $\hat{t} < t_{(k)}^{(1)} < t_k$ and $\{t_k\} \to \hat{t}$, implies that $\{t_m^{(1)}\} \to \hat{t}$ and $x(t_m^{(1)}; \hat{t}, \hat{x}) \to \hat{x}$. Therefore, taking into account the form of the function $\dot{F}$ at the point $(\hat{t}, \hat{x}) \in \bar{B}_{\bar{\varepsilon}_0}$, we obtain

$$(\dot{F}(\hat{t}, \hat{x}) = 0) \Rightarrow \dot{F}(t, x) = 0, \quad \forall (t, x) \in \bar{J}$$

In exactly the same way (repeating the previous reasoning verbatim except for the substitution $F(t,x) \to \dot{F}(t,x)$), we conclude that $\mathcal{D}^{(2)}F(t,x) = 0$, $\forall (t,x) \in \bar{J}$, and so on, all in all $m + 1$ times. This proves the lemma.

The assumptions of Lemma 2 may obviously be weakened, but that is not essential here.

In the autonomous case, any ω-limit set is automatically invariant. In that case Lemma 2 immediately implies the proposition stated in Corollary 2.

In the non-autonomous case, unlike the autonomous case, a limit set need no longer be invariant. In that case, therefore, Lemma 2 does not imply any propositions regarding the limit set.

This is also true in the case of a system (1.1) of class $\mathcal{K}$: There is no implicative relation between Lemmas 1 and 2. However, the case of systems of class $\mathcal{K}$ differs from the general case in that here, nevertheless, it proves possible, as shown above, to achieve an analytical formulation of the equations that must be satisfied by points of the appropriate limit sets. Note that the functions $\mathcal{D}^{(k)}[F](x)$ in Lemma 1 (and Corollary 1) are simply the kth Lie derivatives of the function $F(x)$ along trajectories of the autonomous system $\dot{x}_i = \upsilon_i^*(x)$, where the functions $\upsilon_i^*(x)$ are the limits as $t \to \infty$ of the functions $\upsilon_i(t,x)$ that define the non-autonomous system (1.1) considered in Lemma 1.

In that connection, the following remark may the made. Let the $C_{tx}^{0,1}$-functions $\upsilon_i(t,x)$ defining system (1.1), which has a unique equilibrium position $x_0 = 0$, converge as $t \to +\infty$ uniformly in $\mathbf{R}^n$ to functions that are analytic in $\mathbf{R}^n$, say $\upsilon_i^*(x)$. If at the same time the system $\dot{x}_i = \upsilon_i^*(x)$ does not have invariant sets other than $x_0 = 0$ which are subsets of level sets of an analytic function $F(x)$, then system (1.1) does not have any trajectory whose ω-limit set is distinct from $x_0 = 0$ and is a subset of some level set of $F(x)$.

Indeed, supposing the contrary, one arrives at a contradiction using Lie's formula and taking Corollary 1 into consideration.

We also note that all the propositions of Section 1 remain valid in the case when the limit (or invariant) set consists of only one point.

2. WEAKENING THE CONDITION OF THE SIGN-DEFINITE DERIVATIVE IN LYAPUNOV'S FIRST INSTABILITY THEOREM AND ASYMPTOTIC STABILITY THEOREM, AND IN ANALOGUES OF KRASOVKII'S AND BARBASHIN'S THEOREMS

Lemma 1 may be used to weaken the stipulation made in certain theorems of Lyapunov's second method that the derivative of the function V must be sign-definite.

Throughout what follows, the properties of stability, instability, uniform stability and uniform asymptotic stability being considered (or used) will be understood in the usual sense, that is, in accordance with Lyapunov's definitions [1]. Let us recall them briefly.

The zero equilibrium position $x_0 = 0$ of system (1.1) is said to be stable if, for every arbitrarily small number $\varepsilon > 0$ and every time $t_0 \geqslant 0$, a number $\delta = \delta(\varepsilon, t_0)$ exists, such that, if $|x_0| < \delta(\varepsilon, t_0)$, then $|x(t; t_0, \bar{x}_0)| < \varepsilon$, $\forall t \geqslant t_0$. If the number δ is independent of the time t_0, depending only on the value of ε: $\delta = \delta(\varepsilon)$, the zero equilibrium of system (1.1) is said to be uniformly stable, and in the case when, in addition, some neighbourhood of zero $\mathcal{D} \subset \mathbf{R}^n$ exists, such that

$$\lim_{t \to +\infty} \left|x(t;\, t_0, x^0)\right| \to 0, \ \forall x^{(0)} \in \mathcal{D}, \quad \forall t_0 > 0$$

the solution $x_0 = 0$ is said to be uniformly asymptotically stable.

In turn, if $x_0 = 0$ is the zero equilibrium position of system (1.1) and a pair of numbers $\varepsilon > 0$, $t_0 \geqslant 0$ exists, such that for every $\delta > 0$ a point $x^{(0)}$: $|x^{(0)}| < \delta$ and a number $t > t_0$ exist, such that $|x(t, t_0, x^{(0)})| > \varepsilon$, then the equilibrium $x_0 = 0$ is unstable.

2.1. *Weakening the sign-definiteness condition imposed on the derivative in Lyapunov's first instability theorem*

Proposition 1. Let $x_0 = 0$ be an equilibrium position of system (1.1), and assume that the right-hand sides of the equations satisfy the conditions

$$v_i(t, x) \in C_{tx}^{0,1}(\mathbf{R}^+_{\{t\}} \times \bar{B}_{\varepsilon_0}), \quad v_i(t, x) \underset{t \to +\infty}{\overset{}{\rightrightarrows}}_{\bar{B}_{\varepsilon_0}} v_i^*(x), \quad v_i^*(x) \in C^m(\bar{B}_{\varepsilon_0}), \quad m \geq 0$$

Suppose for system (1.1) that a number $t_0 > 0$ and a number $V(t, x)$ exist:

$$V(t, x) \in C_{tx}^{1,1}(\mathbf{R}^+_{\{t\}} \times B_{\varepsilon_0}), \quad V(t, 0) = 0, \quad \forall t$$

$$V(t, x) \underset{t \to +\infty}{\rightrightarrows}_{B_{\varepsilon_0}} V^*(x), \quad V^*(x) \in C^{m+1}(B_{\varepsilon_0})$$

such that

(1) $\dot{V}(t, x) \geq 0, \forall t \geq t_0, x \in B_{\varepsilon_0}$;

(2) $\forall \delta\, \exists x_{(\delta)}, |x_{(\delta)}| < \delta$: $V(t_0, x_{(\delta)}) > 0$

(3) for any $c > 0$, the algebraic system

$$\{V^*(x) = c,\, \mathcal{D}_*^{(p)}[V^*](x) = 0,\, p = 1, 2, \ldots, m+1\} \tag{2.1}$$

where

$$\mathcal{D}_*^{(k+1)}[\cdot] = \mathcal{D}_*^{(1)}[\mathcal{D}_*^k[\cdot]]$$

has no solutions in some neighbourhood of zero in $\mathbf{R}^n$.

Then the equilibrium $x_0 = 0$ is unstable.

Proof. We shall assume without loss of generality that for no $c > 0$ does system (2.1) have solutions in the domain B_{ε_0}.

Suppose the equilibrium $x_0 = 0$ is stable. Then, first, $\delta > 0$ exists, such that each trajectory $x(t;\, t_0, \hat{x})$, $\hat{x} \in B_{\tilde{\delta}}(0)$ is defined over the entire time axis $\mathbf{R}^+_{\{t\}}$, and, second

$$\forall \varepsilon \exists \delta(\varepsilon, t_0) < \tilde{\delta}: \Rightarrow \left|x(t;\, t_0, \hat{x})\right| < \varepsilon, \quad \forall t \geq t_0, \quad \forall \tilde{x} \in B_\delta \tag{2.2}$$

Now fix some number $\hat{\delta} > 0$, $\hat{\delta} < \delta(\varepsilon_0/2, t_0)$, $(\delta(\varepsilon_0/2, t_0) < \hat{\delta})$. Let $x_{(\hat{\delta})}$ be the point of condition 2 corresponding to the number $\hat{\delta}$.

Then the trajectory $x(t;\, t_0, x_{(\hat{\delta})})$ of system (1.1) may be continued to the entire time axis $\mathbf{R}^+_{\{t\}}$ and

$$x(t;\, t_0, x_{(\hat{\delta})}) \in B_{\varepsilon_0/2}, \quad \forall t \geq t_0 \tag{2.3}$$

Consequently, one can define for that trajectory a non-empty limit set

$$\pi(t_0;\, x_{(\hat{\delta})}) \neq \varnothing, \quad \pi(t_0;\, x_{(\hat{\delta})}) \subset \bar{B}_{\varepsilon_0/2} \subset B_{\varepsilon_0} \tag{2.4}$$

There are two possibilities: the set (2.4) consists of a single point $x = x_1$ (Case 1) or of more than one point (Case 2).

Consider Case 2. Let

$$x_1 \in \pi(t_0; x_{(\hat{\delta})}), \quad x_2 \in \pi(t_0; x_{(\hat{\delta})}), \quad x_1 \neq x_2$$

We have

$$\exists\{t_k^{(l)}\}_{k \to \infty} \to \infty, \quad \{x(t_k^{(l)}; t_0, x_{(\hat{\delta})})\}_{k \to \infty} \to x_l; \quad l = 1, 2 \tag{2.5}$$

Now suppose that

$$V^*(x_1) \neq V^*(x_2), \quad V^*(x_1) > V^*(x_2) \tag{2.6}$$

The uniform convergence condition $V(t, x) \underset{t \to \infty}{\rightrightarrows} V^*(x)$ in the domain B_{ε_0}, and the condition

$$x_1 \in \pi(t_0; x_{(\hat{\delta})}) \subset B_{\varepsilon_0}, \quad x_2 \in \pi(t_0; x_{(\hat{\delta})}) \subset B_{\varepsilon_0}$$

imply, in view of the convergence (2.5), that

$$\{V(t_k^{(l)}; x(t_k^{(l)}; t_0, x_{(\hat{\delta})})\}_{k \to \infty} \to V^*(x_l), \quad l = 1, 2 \tag{2.7}$$

It follows from condition 1, in view of condition (2.3), that from a time $t = t_0$ on, $V(t, x)$ is a non-decreasing function along the trajectory $x(t; t_0, x_{(\hat{\delta})})$:

$$V(\tau_2, x(\tau_2; t_0, x_{(\hat{\delta})})) \geq V(\tau_1, x(\tau_1; t_0, x_{(\hat{\delta})})) \geq V(t_0, x_{(\hat{\delta})}), \quad \forall \tau_2, \tau_1 : \tau_2 \geq \tau_1 \geq t_0 \tag{2.8}$$

At the same time, one can always extract from the sequences $\{t_k^{(l)}\} \to \infty$ subsequences $\{\tilde{t}_m^{(l)}\} \to \infty$ $(l = 1, 2)$ such that

$$t_0^m < \tilde{t}_p^{(1)} < \tilde{t}_p^{(2)} < \tilde{t}_{p+1}^{(1)}, \quad \forall p = 1, 2, \ldots \tag{2.9}$$

Hence, on the basis of inequality (2.8), we have

$$V(\tilde{t}_p^{(2)}; x(\tilde{t}_p^{(2)}; t_0, x_{(\hat{\delta})})) \geq V(\tilde{t}_p^{(1)}; x(\tilde{t}_p^{(1)}; t_0, x_{(\hat{\delta})})), \quad \forall p = 1, 2, \ldots \tag{2.10}$$

Taking the convergences (2.7) into consideration, we obtain

$$\left\{V(\tilde{t}_p^{(l)}); x(\tilde{t}_p^{(l)}; t_0, x_{(\hat{\delta})})\right\}_{p \to \infty} \to V^*(x_l), \quad l = 1, 2 \tag{2.11}$$

Thus, conditions (2.10) and (2.11) imply the inequality $V^*(x_2) \geqslant V^*(x_1)$, contrary to assumption (2.6). Similar reasoning clearly leads to a contradiction from the assumption that $V^*(x_1) > V^*(x_2)$.

As a result, we obtain

$$V^*(x_1) = V^*(x_2), \quad \forall x_1 \in \pi(t_0, x_{(\hat{\delta})}), \quad \forall x_2 \in \pi(t_0, x_{(\hat{\delta})}) \tag{2.12}$$

Now it is also true, by inequality (2.8), that

$$V(t; x(t; t_0, x_{(\hat{\delta})})) \geq V(t_0, x_{(\hat{\delta})}) \stackrel{\text{def}}{=} C_{(t_0, x_{(\hat{\delta})})}, \quad \forall t > t_0$$

where by condition 2 the number $C_{(t_0, x_{(\hat{\delta})})}$ is positive. Therefore, as a consequence of the convergences (2.5) and (2.7), we obtain

$$V^*(x_1) = V^*(x_2) = C_{(t_0, x_{(\hat{\delta})})} \geq C_{(t_0, x_{(\hat{\delta})})} > 0, \quad \forall x_1 \in \pi(t_0, x_{(\hat{\delta})}), \quad \forall x_2 \in \pi(t_0, x_{(\hat{\delta})}) \tag{2.13}$$

Turning now to Case 1, we see that condition (2.12) is always satisfied (automatically), and taking inequality (2.8), condition 2, and the convergence

$$V(t_k; x(t_k; t_0, x_{(\hat{\delta})}))_{k \to \infty} \to V^*(x_1)$$

into consideration, we obtain

$$V^*(x_1) = C^*_{(t_0, x_{(\hat{\delta})})} \geq C_{(t_0, x_{(\hat{\delta})})} > 0$$

Thus, condition (2.13), or the inclusion relation

$$\pi(t_k, x_{(\hat{\delta})}) \subset \{x\colon V^*(x) = C^*_{(t_0, x_{(\hat{\delta})})} > 0\}$$

is satisfied for the set (2.4), irrespective of which of cases 1 or 2 holds. But then the trajectory $x(t; t_0, x_{(\hat{\delta})})$ and its ω-limit set (2.13) satisfy all the assumptions of Corollary 1 of Lemma 1 for $\hat{t} = t_0$, $\hat{x} = x_{(\hat{\delta})}$, $F(x) = V^*(x)$. Consequently, all the points of the set $\pi(t_0, x_{(\hat{\delta})})$ satisfy the algebraic system (2.1) with $C = C^*_{(t_0, x_{(\hat{\delta})})} > 0$. However, $\pi(t_0, x_{(\hat{\delta})}) \subset B_{\varepsilon_0}$ and $\pi(t_0, x_{(\hat{\delta})}) \neq \varnothing$, and therefore, by condition 3, such a set cannot satisfy that system of equations. Thus assumption (2.2) is false, and the equilibrium x_0 is unstable, which it was required to prove.

Proposition 1 corresponds to Lyapunov's first instability theorem. In a sense it is a certain supplement to that theorem in the case of systems (1.1) of class $\mathcal{K}$.

However, Lyapunov's theorem is universal in nature and is applicable in the general case of arbitrary systems (1.1). This is of course no longer true for Proposition 1, in the sense that in the general case of arbitrary systems one cannot formulate an analogue of Proposition 1.

Examples. 1. Consider the system

$$\dot{x}_1 = x_1 + 2x_3 + f_1(x, t)$$

$$\dot{x}_2 = x_2 - 2x_1 + f_2(x, t)$$

$$\dot{x}_3 = -2x_1 + 4x_2 - 4x_3 + f_3(x, t)$$

where the functions $f_i(x, t)$ $(i = 1, 2, 3)$ are uniformly convergent as $t \to +\infty$ in some neighbourhood of the point $x = 0$ to functions $f_i^*(x) \in C^1(0)$, and let $V = (x_1^2 + x_2^2 - x_3^2)/2$.

We have

$$\dot{V} = (x_1 + 2x_3 - x_2)^2 + x_1 f_1 + x_2 f_2 - x_3 f_3$$

If

$$x_1 f_1 + x_2 f_2 - x_3 f_3 = (x_1 + 2x_3 - x_2)^2 g(x, t), \quad g(0, t) = 0, \quad g(x, t) \in \mathbf{C}(R_{\{t\}} \times O)$$

then $\dot{V} \geqslant 0$. Condition 3 of Proposition 1 is also satisfied in this case, since the Jacobian of the transformation $x \to \upsilon^*(x)$ is defined in the relevant neighbourhood of the point $x = 0$ and does not vanish. Hence the equilibrium $x_0 = 0$ is unstable.

2. Consider the following system, which may be categorized as a "critical" case

$$\dot{x}_1 = -x_1 + x_3 + f_1(x, t)$$

$$\dot{x}_2 = -2x_2 + (x_1 - x_2)^2 f_2(x, t)$$

$$\dot{x}_3 = 2x_2 + (x_1 - x_2)^2$$

where

$$f_i(0, t) = 0, \quad f_i(x, t) \underset{0}{\rightrightarrows} f_i^*(x), \quad f_i^*(x) \in C^1(O), \quad i = 1, 2$$

$$f_1(x, t) = o(x^2), \quad f_2(x, t) \in C(R_{\{t\}} \times o)$$

Letting $V = x_2 + x_3$, we have $\dot{V} = (x_1 - x_2)^2[1 + f_2(x, t)]$, and therefore, in some neighbourhood of the point $x_0 = 0$, we have $\dot{V} \geqslant 0$. Condition 3 is also satisfied here, so that, by Proposition 1, the equilibrium $x_0 = 0$ is unstable.

Note that the differentiability of the functions $f^*(x)$, as well as the uniform continuity at $x = 0$ of the functions $g(x, t)$ and $f_2(x, t)$ in the first and second of the above examples, respectively, are essential conditions.

We will now consider a few special cases.

The autonomous case. In the autonomous case Proposition 1 takes the following form.

Proposition 1a. Let $x_0 = 0$ be an equilibrium position of the system

$$\dot{x} = \omega_i(x), \quad i = 1, 2, \ldots, n; \quad \omega_i(x) \in C^m(B_{\varepsilon_0}) \tag{2.14}$$

where $B_{\varepsilon_0} \subset \mathbf{R}^n$ is an open sphere centred at $x_0 = 0$, of radius ε_0. Suppose system (2.14) is such that a function $V(x) \in C^{m+1}(B_{\varepsilon_0})$, $m \geqslant 0$, $V(0) = 0$ exists for which

(1) $\dot{V}(x) \geqslant 0$, $\forall x \in B_{\varepsilon_0}$;
(2) $\forall \delta \exists x_{(\delta)}$, $|x_{(\delta)}| < \delta$: $V(x_{(\delta)}) > 0$;
(3) for any $c > 0$, the algebraic system

$$\{V(x) = c, \mathscr{D}^{(p)}(V)(x) = 0, p = 1, 2, \ldots, m+1\} \tag{2.15}$$

where $\mathscr{D}^k(V)(x)$ is the kth Lie derivative of the function V along trajectories of system (2.14), has no solutions in some neighbourhood of zero in $\mathbf{R}^n$.

Then the equilibrium $x_0 = 0$ is unstable.

Proposition 1a, as a formal corollary of Proposition 1, may naturally be proved directly as well, without reference to Proposition 1, just as Corollary 2 of Lemma 1 may be proved directly, without using the lemma (see Section 1).

Indeed, the stability assumption implies the existence of ω-limit sets $\pi(x_{(\delta)}) \neq \varnothing$, $\pi(x_{(\delta)}) \subset B_{\varepsilon_0}$, (where $x_{(\delta)}$ are the points in condition 2) which, by condition 1, are subsets of certain level sets $\{V(x) = c > 0\}$ of the function V. But then Corollary 2 of Lemma 1 (see Section 1) yields a contradiction to condition 3.

The problem of weakening the condition of Lyapunov's first instability theorem and asymptotic stability theorem that the derivative be sign-definite has been studied for the autonomous and periodic cases by Krasovskii [5]. Proposition 1a is a corollary of an analogue of Krasovskii's instability theorem [5, 7], and Lemma 2 is thus a certain extension of Krasovskii's theorem.

We also mention that Proposition 1a differs from the formulation of Lyapunov's first instability theorem for the autonomous case in that the condition

$$\dot{V}(x) > 0, \quad \forall x \in B_{\varepsilon_0} \tag{2.16}$$

of Lyapunov's theorem is replaced by the combination of condition (2.16) and the condition that the algebraic system (2.15) should have no solutions in the domain B_{ε_0} for any $c > 0$. This latter condition, however, may always be verified directly in each specific case (based on an analysis of the given functions $\omega_i(x)$ and the function V) in an explicit analytical form. Moreover, if $m \geqslant n-1$, it holds almost everywhere.

The degenerate case. Let us consider the degenerate case, in which the function $V^*(x)$ occurring in the assumptions of Proposition 1 vanishes identically. Then condition 3 is automatically satisfied, the convergence condition $\upsilon_i(t, x) \underset{t \to \infty}{\rightarrow} \upsilon_i^*(x)$ as $t \to \infty$ becomes superfluous, and Proposition 1 becomes the following statement.

Suppose a system $\dot{x}_i = \upsilon_i(t, x)$, $(i = 1, \ldots, n)$, having an equilibrium position $x_0 = 0$, is such that for some C^1-function $V(t, x)$, $V(t, 0) = 0$, and some number $t_0 > 0$, the following conditions are satisfied:

(1) $\dot{V} \geqslant 0$, $\forall t \geqslant t_0$, $\forall x \in B_{\varepsilon_0}$;
(2) $\forall \delta \exists x_{(\delta)}$, $|x_{(\delta)}| < \delta$: $V(t_0, x_{(\delta)}) > 0$, where the C-function $V(t, x)$ converges to zero as $t \to \infty$ uniformly in the domain B_{ε_0}.

Then the equilibrium $x_0 = 0$ is unstable.

This proposition is analogous to a previous theorem of Persidskii [4].

2.2. Weakening of the sign-definiteness condition in Lyapunov's asymptotic stability theorem and in the supplement to Krasovskii's theorem

Suppose for some system (1.1) of class $\mathscr{K}$ a function $V(t, x)$ exists which satisfies all the conditions of Proposition 1. In that case each of the trajectories $x(t; t_0, x_{(\delta)})$, $\delta < \varepsilon_0/2$, where $x_{(\delta)}$ is the point in condition 2 of Proposition 1, will at some time $t_{(\delta)} > t_0$ intersect the sphere $|x| = \varepsilon_0/2$ (see above).

If that is the case for any point $x \neq 0$ of the domain $B_{\varepsilon_0/2}$ and any t_0, and if the solutions of system (1.1) are continuable to the entire negative time axis, then, one the basis of the proof of Proposition 1, we may conclude that the motion in system (1.1) is then the reverse of the motion in a system with an uniformly asymptotically stable equilibrium position $x_0 = 0$. (This case is naturally a special instance of the general case of motion with unstable x_0, regarding which Proposition 1 was formulated.)

In other words, if the function $V(t,x)$ in Proposition 1 is positive-definite, then, making the replacement $\geqslant 0 \to \leqslant 0$ in condition 1, we obtain

Proposition 2. Let system (1.1) be a system of class $\mathcal{K}$ for which a number $t_0 > 0$ and a $C^{1,1}_{tx}$-function $V(t,x)$: $(\mathbf{R}_{\{t\}} \times B_{\varepsilon_0}) \to \mathbf{R}$; $V(t, 0) = 0$, $\forall t$ exist, which converges as $t \to 0$, uniformly in the domain B_{ε_0}, to a function $V^*(x) \in C^{m+1}(B_{\varepsilon_0})$ and moreover satisfies the following conditions:

(1) $\dot{V}(t,x) \leqslant 0\ \forall t \geqslant t_0, \forall x \in B_{\varepsilon_0}$;

(2) $V(t,x) \geqslant W(x) > 0\ \forall t \geqslant 0, \forall x \in B_{\varepsilon_0}, x \neq x_0 = 0, W(x_0) = 0$

(3) for any $c > 0$, there is a neighbourhood of $x_0 = 0$ in $\mathbf{R}^n$ in which the algebraic system (2.1) has no solutions.

Then the equilibrium $x_0 = 0$ is uniformly asymptotically stable.

Proof. The method of proof is the same as that used to prove Proposition 1.

As in the proof of Proposition 1, we may assume without loss of generality that the domain occurring in condition 3 contains B_{ε_0}.

By the condition that the function $V(t,x)$ converge uniformly as $t \to +\infty$ in the domain B_{ε_0}, we have

$$\begin{gathered} \forall \varepsilon \exists T(\varepsilon), \delta_1(\varepsilon)\ \forall x\colon |x| < \delta_1(\varepsilon), \quad \forall t > T(\varepsilon) \\ \left|V(t,x) - \lim_{t\to\infty} V(t,0)\right| = |V(t,x)| < \varepsilon \end{gathered} \tag{2.17}$$

By the continuity of the function $V(t,x)$ in the domain $0 \leqslant t, x \in B_{\varepsilon_0}$, thanks to which this function is continuous in x, uniformly in t for $0 \leqslant t \leqslant T(\varepsilon)$ is the number in condition (2.17), we have

$$\begin{gathered} \exists \delta_2(\varepsilon) = \delta_2(T(\varepsilon), \varepsilon) : \forall x : |x| < \delta_2(\varepsilon)\ \forall t : 0 \le t \le T(\varepsilon) \\ |V(t,x) - V(t,0)| = |V(t,x)| < \varepsilon \end{gathered} \tag{2.18}$$

Conditions (2.17) and (2.18) imply

$$\forall \varepsilon\ \exists \delta(\varepsilon) = \min(\delta_1(\varepsilon), \delta_2(\varepsilon)) : \forall x : |x| < \delta(\varepsilon) \Rightarrow |V(t,x)| < \varepsilon, \quad \forall t \ge 0$$

Thus, if the conditions of Proposition 2 are satisfied, the function $V(t,x)$ tends to an infinitesimal upper limit. But then, in view of conditions 1 and 2, it follows at once from Lyapunov's stability theorem that the equilibrium $x_0 = 0$ is uniformly stable. In other words, a number $\tilde{\eta}$ exists such that the trajectory $x(t;\tilde{t},\tilde{x})\ \forall \tilde{x} \in B_{\tilde{\eta}}$ is continuable to the entire time axis $\mathbf{R}^+_{\{t\}}$ and

$$\forall \varepsilon\ \exists \eta(\varepsilon) < \tilde{\eta}, \quad \eta(\varepsilon) < \varepsilon : \forall \tilde{x} : |\tilde{x}| < \eta(\varepsilon), \quad \forall \tilde{t} \ge 0 \quad |x(t;\tilde{t},\tilde{x})| < \varepsilon, \quad \forall t \ge \tilde{t} \tag{2.19}$$

We shall now show that any trajectory of system (1.1) emanating at some instant of time from a point of the sphere

$$B_{\eta(\varepsilon_0/2)} = \{x\colon |x| < \eta(\varepsilon_0/2)\}$$

will tend to the point $x_0 = 0$ as $t \to +\infty$ (here $\eta(\varepsilon)$ is the function defined in (2.19) and $\varepsilon_0 > 0$ is the number occurring in the assumptions of Proposition 2).

Suppose the contrary: a point $\hat{x}$ and a time $\hat{t} \geqslant 0$ exist such that

$$|\hat{x}| < \eta(\varepsilon_0/2) : \exists \hat{\delta} : \forall k > \tilde{t} \quad \exists t_{(k)} > k : |x(t_{(k)}; \hat{t}, \hat{x})| > \hat{\delta} \tag{2.20}$$

Conditions (2.19) and (2.20) taken together give

$$\varepsilon_0/2 > |x(t_{(k)}; \hat{t}, \hat{x})| > \hat{\delta}\ \ \forall k > \hat{t} \tag{2.21}$$

Therefore, the sequence of points $x(t_k; \hat{t}, \hat{x})$, $t_k > k > \hat{t}$ has a partial limit x_1:

$$\begin{gathered} \exists k(m) : \mathbf{N} \to \mathbf{N} : t^{(1)}_m = t_{k(m)}, \ m = 1, \ldots; \quad (k(s) > \hat{t}\ \ \forall s) \\ x(t^{(1)}_m; \hat{t}, \hat{x}) \underset{m\to\infty}{\to} x_1, \quad \varepsilon_0/2 \ge |x_1| \ge \hat{\delta} \end{gathered} \tag{2.22}$$

Since the sequence $\{t_m^{(1)}\}$ satisfies the condition $\{t_m^{(1)}\} \underset{m\to\infty}{\to} \infty$, as a subsequence of the sequence $\{t_k\}_{k\to\infty} \to \infty$, and since each of the points $x(t_m^{(1)}; \hat{t}, \hat{x})$ lies on the semi-trajectory $x(t; \hat{t}, \hat{x})$, $t > \hat{t}$, condition (2.20) means that one can define for the trajectory $x(t; \hat{t}, \hat{x})$ a non-empty ω-limit set

$$\pi(\hat{t}, \hat{x}) \subset \bar{B}_{\varepsilon_0/2} \subset B_{\varepsilon_0}, \quad x_1 \in \pi(\hat{t}, \hat{x}) \tag{2.23}$$

as follows from condition (2.19) in view of the condition $|\hat{x}| < \eta(\varepsilon_0/2)$ in (2.20).

There are two possible cases: the set (2.23) consists of a single point x_1 (Case 1) or of more than one point (Case 2).

Consider Case 2.

Let $x = x_2$ be any point of the set $\pi(\hat{t}, \hat{x})$ distinct from x_1.

We have

$$\exists\{t_m^{(2)}\} \underset{m\to\infty}{\to} \infty: \{x(t_m^{(2)}; \hat{t}, \hat{x})\} \underset{m\to\infty}{\to} x_2 \tag{2.24}$$

Then, reasoning from the convergences (2.20), (2.24), the uniform convergence $V(t,x) \underset{t\to\infty}{\to} V^*(x)$ in the domain B_{ε_0}, the inclusion relation $\pi(\hat{t}, \hat{x}) \subset \bar{B}_{\varepsilon_0/2}$ (2.21), and also from condition 1 in view of the condition $\forall t \geqslant \hat{t} x(t; \hat{t}, \hat{x}) \in B_{\varepsilon_0/2}$, we deduce as in the proof of Proposition 1 that

$$V^*(x_2) = V^*(x_1) \tag{2.25}$$

If Case 1 holds, this equality is automatically true. Hence Eq. (2.25) will always hold, irrespective of which of cases 1 or 2 takes place.

It now follows from the fact that the function $W(x)$ is positive-definite in the domain $\bar{B}_{\varepsilon_0/2}$, in view of the condition $\varepsilon_0/2 > \hat{\delta}$, that

$$\exists \hat{c} > 0: c_0/2 > |x| > \hat{\delta} \Rightarrow W(x) > \hat{c}$$

where $\hat{\delta} > 0$ is the number from (2.20).

Hence, in view of the right-hand inequality of (2.21) and by condition 2,

$$V(t_m^{(1)}; x(t_m^{(1)}; \hat{t}, \hat{x})) \geq W(x(t_m^{(1)}; \hat{t}, \hat{x})) > \hat{c} \tag{2.26}$$

From the condition $V(t,x) \underset{t\to\infty}{\overset{B_{\varepsilon_0}}{\rightrightarrows}} V^*(x)$, taking account of the conditions (2.22) and $\{t_m^{(1)}\} \underset{m\to\infty}{\to} \infty$, we infer

$$\{V(t_m^{(1)}; x(t_m^{(1)}; \hat{x}, \hat{t}))\} \underset{m\to\infty}{\to} V^*(x_1)$$

where x_1 is the point from condition (2.22). But then, by virtue of inequality (2.26), it is necessarily true that

$$V^*(x_1) = c^*_{(\hat{t}, \hat{x})} \geq \hat{c} > 0 \tag{2.27}$$

As a result, we deduce, on the basis of Eq. (2.25) and the inclusion relation (2.23), that for the trajectory $x(t; \hat{t}, \hat{x})$, where $(\hat{t}, \hat{x})$ are the initial data from condition (2.20), and its ω-limit set (2.23), all the conditions of Corollary 1 to Lemma 1 are satisfied. Consequently, all the points of the set (2.23) satisfy algebraic system (2.1) with a number $c = c^*_{(\hat{t}, \hat{x})}$ which is positive by condition (2.27).

But this contradicts condition 3, that is, assumption (2.20) is false. Therefore,

$$x(t; \hat{t}, \hat{x}) \underset{t\to+\infty}{\to} x_0 = 0 \ \forall \hat{x}, |\hat{x}| < \eta(\varepsilon_0/2), \quad \forall \hat{t} \geq 0$$

which, together with condition (2.19), completes the proof of Proposition 2.

In the special case of an autonomous system, Proposition 2 becomes the corollary obtained from Proposition 1a if condition 1 in the latter is replaced by the condition $\dot{V}(x) \leqslant 0$, $\forall x \in B_{\varepsilon_0}$, condition 2 by the condition $V > 0$, $\forall x \in B_{\varepsilon_0}$, $x \neq 0$, and the conclusion of instability by the conclusion that the system is asymptotically stable.

The corollary thus obtained from Proposition 2 for the autonomous case corresponds to Krasovskii's asymptotic stability theorem [5, 7].

Examples. 1. For the system

$$\dot{x}_1 = x_1 - 2x_3 + f_1(x)$$

$$\dot{x}_2 = x_2 + 2x_1 + f_2(x)$$

$$\dot{x}_3 = x_3 - 2x_2 + f_3(x)$$

$$f_i(x) \in C^1(\mathbf{R}^3)$$

and the function $V = (x_1^2 + x_2^2 + x_3^2)/2$, we have

$$\dot{V} = (x_1 + x_2 - x_3)^2 + x_1 f_1 + x_2 f_2 + x_3 f_3$$

If

$$x_1 f_1 + x_2 f_2 + x_3 f_3 = (x_1 + x_2 - x_3)^2 g(x), \quad g(x) \in C(\mathbf{R}^3)$$
$$g(0) = 0$$

then this system necessarily satisfies all the assumptions of Proposition 1a, and the inverse system satisfies all the assumptions of Proposition 2.

2. We will also consider the system describing a Van der Pol pendulum

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 + \mu x_2 (1 - x_1)^2, \quad \mu > 0$$

For the function $V = (x_1^2 + x_2^2)/2$ we have $\dot{V} = \mu x_2^2(1 - x_1^2)$, and therefore, in some neighbourhood of the zero equilibrium position, $\dot{V} \geqslant 0$.

System (2.15) can have solutions only at points with $x_2 = 0$, but in that case the equation

$$\dot{V}|_{x_2 = 0} = 2\mu x_1^2 (1 - x_1^2) = 0$$

will have (in a sufficiently small neighbourhood of $x_0 = 0$) a unique solution $x_1 = 0$. Hence $x_0 = 0$ is the only solution of system (2.15), and the equilibrium x_0 is unstable.

For the inverse system

$$\dot{x}_1 = -x_2, \quad \dot{x}_2 = x_1 - \mu x_2 (1 - x_1)^2$$

and the function $V = (x_1^2 + x_2^2)/2$, as in the previous example, all the assumptions of Proposition 2 hold, and the zero equilibrium position is asymptotically stable.

2.3. Concerning a supplement to Barbashin's theorem

We will now consider yet another special case in which all the assumptions of Proposition 2 for the functions $\upsilon_i(t, x)$ and some function $V(t, x)$ hold throughout the entire space $\mathbf{R}^n$, while the function $W(x)$ condition 2 satisfies the condition $W(x) \to +\infty$ as $|x| \to 8$.

The following remark, which extends a well-known result of Barbashin [7], holds.

Let $x_0 = 0$ be an equilibrium position of system (1.1), where the right-hand sides of the latter are functions of class $C_{tx}^{0,1}(\mathbf{R}_{\{t\}}^+ \times \mathbf{R}^n)$, and let the functions $\upsilon_i(t, x)$ $(i = 1, \ldots, n)$ converge as $t \to +\infty$ uniformly in $\mathbf{R}^n$ to the functions $\upsilon_i^*(x) \in C^m(\mathbf{R}^n)$. Suppose moreover that a function $V(t, x) \in C_{tx}^{1,1}\{\mathbf{R}_{\{t\}}^+ \times \mathbf{R}^n)\}$ exists which converges at $t \to +\infty$ uniformly in $\mathbf{R}^n$ to a function $V^*(x) \in C^{m+1}(\mathbf{R}^n)$ and satisfies the following conditions:

(1) $\dot{V}(t, x) \leqslant 0$, $\forall t \geqslant 0$, $\forall x \in \mathbf{R}^n$;

(2) $V(t, x) \geqslant W(x) > 0$, $\forall t$, $\forall x \in \mathbf{R}^n$, $x \neq x_0$; $W(x_0) = 0$; $W(x) \to +\infty$;

(3) the algebraic system (2.1) has no solutions in $\mathbf{R}^n$ for any $c > 0$.

Then the equilibrium $x_0 = 0$ is asymptotically stable in the large [5, 7].

We recall the zero equilibrium position $x_0 = 0$ of system (1.1) is said to be asymptotically stable in the large if it is asymptotically stable and the solution $x(t; t_0, x^{(0)})$ with any initial data $(t_0, x^{(0)})$ in the phase space $\mathbf{R}_{\{t\}}^+ \times \mathbf{R}^n$ of system (1.1) tends to the equilibrium $x_0 = 0$ as $t \to +\infty$.

Indeed, take any point of (t, x)-space, $x(\hat{t}, \hat{x})$, $\hat{x} \neq x_0$. By the last part of condition 2

$$\exists \mathcal{R} = \mathcal{R}(V(\hat{t}, \hat{x})) \colon W(x) > V(\hat{t}, \hat{x}), \quad \forall x \; |x| \geq \mathcal{R}(V(\hat{t}, \hat{x}))$$

Hence, in view of condition 1 and the first part of condition 2,

$$|x(t;\,\hat{t},\hat{x})| < \mathfrak{R}(V(\hat{t},\hat{x})), \quad \forall t \geq \hat{t}$$

Thus, every semi-trajectory $x(t;\,\hat{t},\,\hat{x})$ of system (1.1) will remain within some sphere $B_{\mathfrak{R}(V(\hat{t},\hat{x})}$ and therefore has a non-empty ω-limit set $\pi(\hat{t},\,\hat{x})$. Reasoning now exactly as in the proof of Proposition 2 (from the assumption (2.20) until the end), one obtains the desired conclusion.

For example, for the system inverse to that presented above in Example 1, the equilibrium $x_0 = 0$ is asymptotically stable in the large if

$$x_1f_1 + x_2f_2 + x_3f_3 = (x_1+x_2-x_3)^2 g(x); \quad g \geq 0, \quad \forall x \in \mathbf{R}^n$$
$$g(0) = 0$$

3. WEAKENING THE SIGN-DEFINITENESS CONDITION FOR THE DERIVATIVE IN THEOREMS ANALOGOUS TO CHETAYEV'S AND PERSIDSKII'S THEOREMS

We note that Lemma 1 may used to weaken the condition that the derivative $\dot{V}$ be sign-definite for system class $\mathcal{K}$, not only in the theorems of Lyapunov's second method in which this condition is required to hold for all points in some neighbourhood of $x_0 = 0$, but also for those where this condition must hold in some domain ω, $x_0 \in \bar{\omega}$, which is not a complete neighbourhood of zero.

3.1. A supplement of Chetayev's theorem

Proposition 3.1. Suppose that, for some system (1.1) of class $\mathcal{K}$, number $t_0 > 0$ and a C^{-1}-function $V(t,x)$: $\mathbf{R}^+_{\{t\}} \times B_{\varepsilon_0} \to \mathbf{R}$, $V(t, 0) = 0$, $\forall t$, which converges as $t \to +\infty$ uniformly in the domain B_{ε_0} to a function $V^*(x) \in C^{m+1}(B_{\varepsilon_0})$, exist such that

(1) $\forall \tilde{t} \geqslant t_0$, $\forall \tilde{x} \in B_{\varepsilon_0}$, $\tilde{x} \in \{x: V(\tilde{t},x) > 0\} \Rightarrow \dot{V}(\tilde{t},\tilde{x}) \geqslant 0$;

(2) $\forall\delta\exists_{x(\delta)}$, $|x_{(\delta)}| < \delta$; $V(t_0, x_{(\delta)}) > 0$;

(3) for any $c > 0$, there is a neighbourhood of zero in $\mathbf{R}^n$ in which the algebraic system (2.1) has no solutions.

Then the equilibrium $x_0 = 0$ is unstable.

Proof. Suppose the equilibrium $x_0 = 0$ is stable. Then, on the basis of the conditions of Proposition 3.1, it is obvious that all the arguments in the proof of Proposition 1, from the very beginning up until formula (2.7) inclusive, remain valid in this case also.

We shall now show that here, just as under the assumptions of Chetayev's theorem, the function $V(t, x)$, defined in the proof of Proposition 1 along the semi-trajectory $x(t;\, t_0, x_{(\hat{\delta})})$, $t \geqslant t_0$, is a non-decreasing function.

We have

$$V(t;\, x(t;\, t_0, x_{(\hat{\delta})})) = V(t_0, x_{(\hat{\delta})}) + \int_{t_0}^{t} \dot{V}(\tau;\, x(\tau;\, t_0, x_{(\delta)})d\tau, \ \forall t \geq t_0 \tag{3.1}$$

If follows from condition 2 with $t = t_0$ that $V(t_0, x_{(\delta)}) > 0$. Let us assume now that this condition fails to hold at some point of the semi-trajectory $x(t;\, t_0, x_{(\hat{\delta})})$:

$$\exists t > t_0\colon\ V(t;\, x(t;\, t_0, x_{(\hat{\delta})})) \leq 0 \tag{3.2}$$

But the function $V(t, x)$ is continuous is the domain $\bar{\mathbf{R}}^+_{\{t\}} \times B_{\varepsilon_0}$ and, since by condition (2.3) the whole semi-trajectory $(t, x(t;\, t_0, x_{(\hat{\delta})}))$, $t \geqslant t_0$ lies within that domain, the function $V(t, x)$ is continuous on the curve $x(t;\, t_0, x_{(\hat{\delta})})$, $t \geqslant t_0$, in $\mathbf{R}^n$. Hence, in view of the condition $V(t_0, x_{(\delta)}) > 0$ and our assumption (3.2), it follows that

$$\exists t^{(0)} > t_0\colon\ V(t^{(0)};\, x(t^{(0)};\, t_0, x_{(\hat{\delta})})) = 0, \quad V(t;\, x(t;\, t_0, x_{(\hat{\delta})})) > 0, \quad \forall t: t_0 \leq t < t^{(0)} \tag{3.3}$$

On the basis of relation (3.1), condition 2 and the first relation of (3.3), we conclude that the segment of the trajectory $(t, x(t; t_0, x_{(\hat{\delta})}))$ and $t_0 \leqslant t \leqslant t^{(0)}$ contains a point $(\tilde{t}, x(\tilde{t}; t_0, x_{(\delta)}))$, $t_0 < \tilde{t} < t^{(0)}$ at which the function $\dot{V}(t, x)$ takes a negative values:

$$\exists \tilde{t} : t_0 < \tilde{t} < t^{(0)} : \dot{V}(\tilde{t}; x(\tilde{t}; t_0, x_{(\hat{\delta})})) < 0 \tag{3.4}$$

But by the second part of condition (3.3) we have

$$V(\tilde{t}; x(\tilde{t}; t_0, x_{(\hat{\delta})})) > 0 \tag{3.5}$$

But then, on the basis of (2.3), (3.4) and (3.5), we arrive at a proposition contrary to condition 1.

Thus assumption (3.2) must be false. Therefore,

$$V(t; x(t; t_0, x_{(\hat{\delta})})) > 0, \quad \forall t \geq t_0$$

whence, in view of the inclusive relation (2.3) and condition 1, it follows that

$$\dot{V}(t, x)|_{V(t;\, x(t;\, t_0,\, x_{(\hat{\delta})}))} \geq 0, \quad \forall t \geq t_0$$

This means that $V(t, x)$ is a non-decreasing function along the semi-trajectory $x(t; t_0, x_{(\hat{\delta})})$, $t \geqslant t_0$. But then, as is really seen on the basis of the assumptions of Proposition 3, the whole proof of Proposition 1, beginning with formula (2.8) inclusive to the end, can be repeated here verbatim.

Thus, the stability assumption is false and the equilibrium $x_0 = 0$ is unstable, which it was required to prove.

Remark. In the above proof it was implicitly assumed that, for every value of $t = \hat{t} \geqslant t_0$,

$$\{x\colon V(\hat{t}, x) > 0, |x| < \varepsilon_0\} \neq \varnothing$$

But this condition was not stipulated in the statement of Proposition 3.1. The fact is, however, that it is not necessary. Indeed, otherwise, as follows from obvious arguments similar to those used in the proof of Proposition 3.1, it would follow from the assumptions that the trajectory $x(t; t_0, x_{(\hat{\delta})})$ cannot be continued for every $\hat{\delta} < \delta(\varepsilon_0/2, t_0)$ to the entire time axis $\mathbf{R}^+_{\{t\}}$. But this means that the equilibrium $x_0 = 0$ is unstable.

Proposition 3.1 is a certain extension of Chetayev's theorem to systems of class $\mathcal{K}$.

The autonomous case. In the autonomous case Proposition 3 becomes the following

Proposition 3.1a. Let $x_0 = 0$ be an equilibrium position of the system. Suppose for system (2.14) that function $V(x) \in C^{m+1}(B_{\varepsilon_0})$, $m \geqslant 0$, $V(0) = 0$ exists, such that, everywhere in a domain $\{x\colon V(x) > 0, |x| < \varepsilon_0\}$ such that the point x_0 is on the boundary of the domain, the function $\dot{V}$ is non-negative, while, for any $c > 0$, there is a neighbourhood of zero in which the algebraic system (2.15) has no solutions. Then the equilibrium $x_0 = 0$ is unstable.

The relationship between Proposition 3.1a and Chetayev's theorem for the autonomous case is the same as between Proposition 1.1a and Lyapunov's first instability theorem for the autonomous case.

For example, for every system of the form

$$\dot{x}_i = f_i(x), \quad x \in \mathbf{R}^n, \quad f_i(0) = 0 \ \forall i = 1, 2, \ldots, n, \quad f_n(x)|_{x_n \geq 0} \geq 0,$$
$$f_i(x) \in C^n(\mathbf{R}^n) \ \forall i = 1, \ldots, n$$

which moreover has the property that the algebraic system $\{\mathcal{D}^1 f_n(x) = \ldots = \mathcal{D}^n f_n(x) = 0\}$ has a unique solution $x = 0$, all the assumptions of Proposition 3.1 hold for the function $V = x_n$, and the zero equilibrium positions of all such systems are unstable.

3.2. A supplement to Persidskii's theorem

Clearly, Lemma 1 may also be used to weaken the condition that the derivative $\dot{V}$ be sign-definite in other theorems of the second method.

In particular, this may be done for those theorems in which it is assumed that a certain sector ω exists (where the term "sector" is understood henceforth as a domain in the sense defined by Persidskii [6]). For example, we have the following

Proposition 3.2. Let system (2.14) have an equilibrium position $x_0 = 0$.

Suppose for system (2.14) that a sector ω [6], $x_0 \in \bar{\omega}$, exists defined in the domain B_{ε_0}, and a function $V(x) \in C^{m+1}(B_{\varepsilon_0})$, $m \geqslant 0$, $V(0) = 0$, such that

(1) $\dot{V}(x) \geqslant 0\ \forall x \in \omega, x \in B_{\varepsilon_0}$;

(2) $\forall\delta\exists x_{(\delta)}, |x_{(\delta)}| < \delta, x_{(\delta)} \in \omega$: $V(x_{(\delta)}) > 0$;

(3) for any $c > 0$, the algebraic system (2.15) has no solutions in the domain $\{x: |x| < \varepsilon_0, x \in \bar{\omega}\} \subset \mathbf{R}^n\}$.

Then the equilibrium $x_0 = 0$ is unstable.

Proof. Suppose the contrary. Then a number $\delta_0 > 0$ exists such that

$$\forall x \in B_{\delta_0}\ |g^t(x)| < \varepsilon_0/2, \quad \forall t \geq 0$$

where $g_t(x)$ is the phase flow of system (2.14) and ε_0 is the number in the condition of Proposition 3.2.

Hence, for every point $x_{(\delta)}$ occurring in condition 2 we have, if $\delta < \delta_0$,

$$|g^t(x_{(\delta)})| < \varepsilon_0/2, \quad \forall t \geq 0 \tag{3.6}$$

Now fix some number $\delta = \hat{\delta} < \delta_0$ and consider the trajectory $g^t(x_{(\hat{\delta})})$, $t \geqslant 0$. By condition (3.6), for the trajectory $g^t(x_{(\hat{\delta})})$ the following ω-limit set exists

$$\pi(x_{(\hat{\delta})}) \neq \varnothing, \quad \pi(x_{(\hat{\delta})}) \subset \bar{B}_{\varepsilon_0/2} \tag{3.7}$$

Based on the definition of the sector ω [4], we conclude that any trajectory emanating at $t = 0$ from an interior point of the domain

$$\{x: x \in \omega,\ |x| < \varepsilon_0\}$$

may leave the domain only through the boundary $|x| = \varepsilon_0$. But by condition (3.6) this is impossible for the trajectory $g^t(x_{(\hat{\delta})})$ (see condition 2). We therefore have

$$g^t(x_{(\hat{\delta})}) \in \omega, \ \forall t \geq 0 \tag{3.8}$$

We now conclude from conditions (3.6) and (3.8), by virtue of condition 1 of our proposition, that along the trajectory $g^t(x_{(\hat{\delta})})$, $t \geqslant 0$, the function $V(x)$ is a non-decreasing function. But then it follows from the definition of a limit set, the continuity of the function $V(x)$ in the domain B_{ε_0}, and condition (3.6) that

$$\forall x_1 \in \pi(x_{(\hat{\delta})}), \quad x_2 \in \pi(x_{(\hat{\delta})}), \quad V(x_1) = V(x_2), \quad \forall t \geq 0 \tag{3.9}$$

Therefore, by inclusion (3.7) and Eq. (3.9), all the assumptions of Corollary 2 to Lemma 1 hold for the limit set $\pi(x_{(\hat{\delta})})$ of the trajectory $g^t(x_{(\hat{\delta})})$ of system (2.14). Thus, all the points of the set $\pi(x_{(\hat{\delta})})$ satisfy system (2.15) with some $c \geqslant V(x_{(\hat{\delta})}) > 0$ and moreover, by condition (3.7) and (3.8), they belong to the domain $\{x: x \in \bar{\omega},\ |x| < \varepsilon_0\}$. But this contradicts condition 3, so that the assumption is false, and the equilibrium position x_0 is unstable, which it was required to prove.

For example, in the system $\dot{x}_1 = x_2, \dot{x}_2 = -x_2 + ax_1^2$, $a > 0$, the instability of the equilibrium position $x_0 = 0$ follows from Proposition 3.2 with sector $\omega = \{x_2 > 0\}$ and the function $V = x_1$.

Proposition 3.2 may also be extended to the non-autonomous case for class $\mathcal{K}$ of systems (1.1).

In conclusion, we note that "the sphere B_{ε_0}" in all the propositions formulated above may naturally be replaced by "some neighbourhood to zero in $\mathbf{R}^n$" (this is unimportant).

The condition $v_i(t, x) \in C^{0,1}_{tx}(\mathbf{R}^+_{\{t\}} \times B_{\varepsilon_0})$ may also be replaced throughout the text by the condition

$$v_i(t, x) \in C(\mathbf{R}^+_{\{t\}} \times B_{\varepsilon_0}); \quad v_i(t, x) \in \mathrm{Lip}_x(L), \quad i = 1, \ldots, n$$

In Propositions 1, 3 and 3.2 the condition $V(t, x) \in C^{1,1}_{tx}(\mathbf{R}^+_{\{t\}} \times B_{\varepsilon_0})$ may be weakened to $V(t, x) \in C(\mathbf{R}^+_{\{t\}} \times B_{\varepsilon_0})$, together with the condition that the function $\dot{V}(t, x)$ be defined everywhere in the domain $\mathbf{R}^+_{\{t\}} \times B_{\varepsilon_0}$.

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